$$I(w^i) = \int_{V_0} U d\tau_0 - \int_{S_\sigma} P_i w^i d\sigma - \int_{V_0} F_i w^i d\tau_0 + \int_{\Omega} \Phi d\omega$$

where U is the elastic energy density of the massive body, $d\tau_0$ is the volume element of the domain V_0 and $d\sigma, d\omega$ are the area elements of the surfaces S_σ and Ω, respectively. This enables us to use the well-developed variational-difference method for a numerical investigation of a broad class of problems.

REFERENCES

1. GALIN L.A., Contact Problems of the Theories of Elasticity and Viscoelasticity. Nauka, Moscow, 1980.
2. PRAGER W. and SYNGE J.L., Approximation in elasticity based on the concept of function space, Q. Appl. Math. 5, 3, 1947.
3. ALEKSANDROV V.M. and MKHITARYAN S.M., Contact Problems for Bodies with Thin Coatings and Interlayers. Nauka, Moscow, 1983.
4. POPOV G.YA., Elastic Stress Concentration Near Stamps, Slits, Thin Inclusions, and Reinforcements. Nauka, Moscow, 1982.
5. CHERNYKH K.F., Linear Theory of Shells, Pt. 2, Leningrad State University, 1964.
6. SAVIN G.N. and TUL'CHII V.I., Plates Reinforced by Steel Rings and Elastic Plates, Naukova Dumka, Kiev, 1971.
7. GUZ A.N., LUGOVOI P.Z. and SHUL'GA N.A., Conical Shells Weakened by Holes. Naukova Dumka, Kiev, 1976.
8. VLASOV V.Z. and LEONT'YEV N.I., Beams, Slabs, and Shells on an Elastic Base. Fizmatgiz, Moscow, 1960.
9. ROZIN L.A., Variational Formations of Problems for Elastic Systems. Leningrad State University, Leningrad, 1978.

Translated by M.D.F.

PMM U.S.S.R., Vol.53, No.4, pp.546-550, 1989
Printed in Great Britain

0021-8928/89 \$10.00+0.00

ON A METHOD OF INVESTIGATING FIBRE STABILITY IN AN ELASTIC SEMI-INFINITE MATRIX NEAR A FREE SURFACE*

A.N. GUZ and YU.N. LAPUSTA

The properties of an infinite characteristic determinant in the three-dimensional linearized problem /1/ of fibre stability in an elastic semi-infinite matrix near a free surface are investigated. As in the case of two and a number of doubly-periodic systems of fibres in an infinite matrix /2-4/ it is proved that the mentioned determinant is a determinant of normal type. Non-linearly elastic transversely isotropic compressible materials are examined within the framework of the theory of finite subcritical deformations without taking account of the specific form of the elastic potential. The results elucidated hold even for different modifications of the theory of small subcritical deformations. Results of an analysis of kindred questions of the theory of elastic wave diffraction are used /5/.

1. *Formulation of the problem.* The characteristic equation. We consider the stability

*Prikl. Matem. Mekhan., 53, 4, 693-697, 1989

of a fibre in a semi-infinite matrix near a free surface. We introduce the following fundamental assumptions: 1) there are no body forces and the semi-infinite body under consideration is loaded by "dead" compressive forces along the fibre axis in such a manner that the shortened fibre and matrix are equal in this direction; 2) rigid contact is made between the fibre and the matrix; 3) the plane matrix surface is force-free; 4) the subcritical state is homogeneous (this assumption is given a foundation by the results in /6/).

Because of assumption 1) the sufficient conditions for the static method (Euler's method) to be applicable for investigating the stability problem under consideration are satisfied. Consequently, we will solve this problem by Euler's method.

We refer the fibre and matrix to Lagrange coordinates that coincide with the deformation with rectangular (x, y, z) and cylindrical (r, θ, x_3) coordinate systems. The connection between these coordinate systems is described by the relationships

$$x = -r\sin\theta,\ y = h - r\cos\theta,\ z = x_3 \tag{1.1}$$

Let the fibre and matrix occupy the respective domains

$$D^{(1)} = \{(r, \theta, x_3) \in \mathrm{R}^+\times\Pi\times\mathrm{R} : r \leqslant R\},\ D = \{(r, \theta, x_3) \in \mathrm{R}^+\times\Pi\times\mathrm{R} : r \geqslant R,\ r\cos\theta \leqslant h\};\ \mathrm{R}^+ = [0, \infty),\ \Pi = [0, 2\pi),\ \mathrm{R} = (-\infty, \infty)$$

We consider the fibre to be strictly within a semi-infinite matrix, in which connection we require satisfaction of the condition

$$R/h < 1 \tag{1.2}$$

Using the notation and following the procedure in /1/, we obtain an infinite homogeneous system of algebraic equations

$$A_{\alpha m}X_m + B_{\alpha m}X_m^{(1)} + \sum_{n=0}^{\infty} M_{\alpha mn}X_n = 0 \quad (\alpha = 1, 2;\ m = 0, 1, 2, \ldots) \tag{1.3}$$

The elements of the matrix $A_{\alpha m}, B_{\alpha m}, M_{\alpha mn}$ are determined from the formulas

$$A_{\alpha mij} = T_{1\alpha ij}(m) K_{m+1}(\zeta_i\gamma R) K_m^{-1}(\zeta_i\gamma R) + T_{2\alpha ij}(m)$$
$$B_{\alpha mij} = T_{1\alpha ij}^{(1)}(m) I_{m+1}(\zeta_i^{(1)}\gamma R) I_m^{-1}(\zeta_i^{(1)}\gamma R) - T_{2\alpha ij}^{(1)}(m)$$

$$M_{\alpha mnij} = \sum_{t=1}^{3} [-T_{1\alpha it}(m) I_{m+1}(\zeta_t\gamma R) + T_{2\alpha it}(m) I_m(\zeta_t\gamma R)] P_{mntj}K_n^{-1}(\zeta_j\gamma R) \tag{1.4}$$
$$P_{mnij} = \varepsilon_m \int_{-\infty}^{\infty} B_{ij}(t)\exp[-\gamma h(R_{jj} + R_{ij})]\exp(-nt) Q_{mij}^{\mp}\, dt$$
$$Q_{mij}^{\mp} = [(R_{ij} + \alpha_j)^{-m} \mp (R_{ij} - \alpha_j)^{-m}]\zeta_i^{\,m}, \quad \gamma = \pi l^{-1}$$

Here l is the buckling half-wavelength; the minus superscript corresponds to $i = 1$ and the plus subscript to $i = 2,3$; $\varepsilon_m = 1/2$ for $m = 0$ and $\varepsilon_m = 1$ for $m > 0$. The functions $B_{ij}(t)$ are determined from the system of equations

$$\left(\alpha_j^2 + \frac{1}{2}\zeta_1^2\right)\left(B_{1j} + \frac{1}{2}\delta_{1j}\right) - \alpha_j\sum_{s=2}^{3} R_{sj}\left(B_{sj} - \frac{1}{2}\delta_{sj}\right) = 0 \tag{1.5}$$
$$\alpha_j\left(B_{1j} + \frac{1}{2}\delta_{1j}\right) - \sum_{s=2}^{3} K_{1s}R_{sj}\left(B_{sj} - \frac{1}{2}\delta_{sj}\right) = 0$$
$$\alpha_j R_{1j}\left(B_{1j} - \frac{1}{2}\delta_{1j}\right) - \sum_{s=2}^{3}(K_{2s} + \alpha_j^2)\left(B_{sj} + \frac{1}{2}\delta_{sj}\right) = 0$$

Here

$$\alpha_j = \zeta_j\,\mathrm{sh}\,t, \quad R_{ij} = \sqrt{\zeta_i^{\,2} + \alpha_j^{\,2}} \quad (i, j = 1, 2, 3)$$
$$K_{1s} = (a_{11}\zeta_s^{\,2} + a_{13} - \sigma_{33}^{*\circ}\lambda_1^{-2})(a_{13} + G_{13})^{-1}$$
$$K_{2s} = {}^1/_2(a_{11}G_{13}\zeta_s^{\,2} + a_{13}G_{13} + a_{13}\sigma_{33}^{*\circ}\lambda_1^{-2})G_{12}^{-1}(a_{13} + G_{13})^{-1}$$

The quantities a_{ij}, G_{1k} which characterize the model of a deformable body, and the quantities ζ_i^2 which depend on the loading conditions and the form of the elastic potential can be determined from formulas in /3, 7/, λ_1 denotes the elongation and $\sigma_{33}^{*\circ}$ is the generalized stress tensor component. Appropriate quantities for the fibre are marked with the superscript (1).

For $m > 0$ the quantities $T_{\beta\alpha ij}(m)$ are determined from the formulas

$$
\begin{gathered}
T_{1111} = T_{112s} = T_{1131} = T_{113s} = T_{2131} = T_{1231} = 0 \\
T_{111s} = \zeta_s \varkappa, \ T_{1121} = -\zeta_1 \varkappa \\
T_{2111} = -T_{211s} = T_{2121} = T_{212s} = m, \ T_{213s} = \lambda_1 (K_{1s} - 1) \\
-\zeta_1^{-1} T_{1211} = \zeta_s^{-1} m T_{121s} = -\zeta_1^{-1} m T_{1221} = \zeta_s^{-1} T_{122s} = \varkappa (m-1)^{-1} T_{2211} = \\
-\varkappa (m-1)^{-1} T_{222s} = 2\lambda_1^2 G_{12} \varkappa^{-1} m \\
-\zeta_s^{-1} \varkappa^{-1} T_{123s} = K_{1s} m^{-1} T_{2231} = m^{-1} T_{223s} = \lambda_1 G_{13} K_{1s} \varkappa^{-1} \\
T_{221s} = 2\lambda_1^2 G_{12} [K_{2s} + \varkappa^{-2} m (m-1)] \\
T_{2221} = -\lambda_1^2 G_{12} [\zeta_1^2 + 2\varkappa^{-2} m (m-1)], \quad \varkappa = \gamma R
\end{gathered} \tag{1.6}
$$

The values of $T_{\beta\alpha\nu k}(0)$ are obtained from $T_{\beta\alpha\nu r}(u)$ for $u = 0$ $(r = k + 1;\ \alpha, \beta, \nu, k = 1, 2;\ s, r = 2, 3)$. In order to obtain expressions for $T^{(1)}_{\beta\alpha ij}(m)$, it is necessary to mark quantities which depend on the subcritical state and the model of the deformable body in the appropriate formulas for $T_{\beta\alpha ij}(m)$ by the superscript (1) and to multiply the expressions obtained by $(-1)^{\beta}$

Let us note that satisfaction of the conditions

$$
\begin{gathered}
\det B_{2m} \neq 0, \quad \det H_m \neq 0 \\
(H_m = (A_{1m} - B_{1m} B_{2m}^{-1} A_{2m}))
\end{gathered} \tag{1.7}
$$

can be required in the problem under consideration without loss of generality.

Indeed, the fact that the first determinant in (1.7) equals zero corresponds to one of the instability modes of an infinite cylinder with a free side surface subjected to a compressive load, while the fact that the second equals zero is one of the fibre buckling modes in an infinite matrix.

We reduce system (1.3) to canonical form

$$
X_m + \sum_{n=0}^{\infty} S_{mn} X_n = 0 \tag{1.8}
$$

$$
S_{mn} = H_m^{-1} (M_{1mn} - B_{1m} B_{2m}^{-1} M_{2mn}) \tag{1.9}
$$

We derive the characteristic equation

$$
\Delta(\varepsilon, \gamma R) = 0 \tag{1.10}
$$

from the conditions for non-trivial solutions to exist, where $\Delta(\varepsilon, \gamma R)$ is the determinant of the infinite homogeneous system of algebraic Eqs.(1.8), and ε is the shortening of the fibre and the matrix.

2. *Certain properties of the system of Eqs.(1.5) and their solution.* We use the following notation: $\delta_j(t)$ is the determinant of the j-th system of Eqs.(1.5) while $\Delta_{ij}(t)$ is the determinant obtained from $\delta_j(t)$ by replacing the i-th column by a column of free terms. Plane and spatial instability problems for a free plane boundary of a semi-infinite body under compression in directions parallel to the free surface are examined in /3, 4/. The characteristic determinants obtained for spatial problems have a structure analogous to $\delta_j(t)$. This enables us to establish the physical meaning of the values t_0 for which $\delta_j(t_0) = 0$: they govern the buckling mode of the plane boundary of the semi-infinite matrix compression along the z axis with half-wavelengths l_z and l_x whose ratio equals $|\alpha_j(t_0)|$.

It follows from the results in /3, 4/ and reasoning of a physical nature that the bonding element (more rigid than the binder) is unstable in the matrix for smaller values of the shortening than the free plane surface is. Consequently, without loss of generality it can later be required that the conditions

$$
\forall t \in (-\infty, \infty), \quad \delta_j(t) \neq 0 \quad (j = 1, 2, 3) \tag{2.1}
$$

be satisfied.

These relationships afford the possibility of determining the unknown functions $B_{ij}(t)$ from the formulas

$$
B_{ij}(t) = \Delta_{ij}(t)\, \delta_j^{-1}(t) \quad (i, j = 1, 2, 3) \tag{2.2}
$$

We introduce $t_1 > 0$ into the consideration such that

$$
\alpha_j^2(t_1) > \max_{i=1,2,3} \zeta_i^2 \quad (j = 1, 2, 3) \tag{2.3}
$$

Because of the continuity of the desired functions $B_{ij}(t)$ that are bounded in any bounded closed segment, including even in $[-t_1, t_1]$

$$\forall t \in [-t_1, t_1], \quad |B_{ij}(t)| < M_1 \tag{2.4}$$

Here and henceforth M_k $(k = 1, 2, 3, \ldots)$ are certain positive constants.

We will now examine the case when $|t| > t_1$. It is seen that the functions $\alpha_j^{-5}(t)\,\Delta_{ij}(t)$ are bounded in $(-\infty, -t_1) \cup (t_1, \infty)$

$$\forall t \in (-\infty, -t_1) \cup (t_1, \infty), \quad |\alpha_j^{-5}(t)\,\Delta_{ij}(t)| < M_2 \tag{2.5}$$

Furthermore, expanding $\alpha_j^{-5}(t)\,\delta_j(t)$ in a power series in $\alpha_j^{-2}(t)$ for $|t| > t_1$, we arrive at the estimate

$$\forall t \in (-\infty, -t_1) \cup (t_1, \infty), \quad |\delta_j(t)| > M_3 |\alpha_j(t)|^{5-2q}, \quad q \in N \tag{2.6}$$

It follows from a combined examination of the estimates (2.5) and (2.6) and formula (2.2) that

$$\forall t \in (-\infty, -t_1) \cup (t_1, \infty), \quad |B_{ij}(t)| < M_4 \alpha_j^{2q}(t) \tag{2.7}$$

Taking into account that

$$|\alpha_j(t)| < \zeta_j \exp|t| \tag{2.8}$$

we obtain from (2.4) and (2.7)

$$\forall t \in \mathbf{R}, \quad |B_{ij}(t)| < M_5 \exp 2q|t| \tag{2.9}$$

Note that because of the evenness of the functions $\delta_j(t)$ the functions $B_{ij}(t)$ have the same evenness as $\Delta_{ij}(t)$. The functions with subscripts 11, 22, 23, 32, 33 are even here while the rest are odd.

3. The basis for the method of reduction to be applicable to the solution of the characteristic equation. Eq. (1.10), on whose left-hand side there is an infinite determinant, has a quite complex structure. Consequently, its analytic solution is not possible. For the numerical solution the characteristic determinant must be replaced by a finite determinant. To justify the replacement we examine certain properties of the determinant $\Delta(\varepsilon, \gamma R)$ and we prove that it is a determinant of normal type

$$\exists M_6, \; S = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{i=1}^{i'} \sum_{j=1}^{j'} |S_{mnij}| < M_6 \tag{3.1}$$

$$i' = \begin{cases} 2, & m = 0 \\ 3, & m > 0 \end{cases}, \quad j' = \begin{cases} 2, & n = 0 \\ 3, & n > 0 \end{cases}$$

Elements of the matrices

$$H_m^{-1} \text{ and } H_m^{-1} B_{1m} B_{2m}^{-1}.$$

occur as cofactors in the expressions for the elements S_{mnij} of the characteristic determinant.

Taking into account that equivalence relationships in the form

$$I_n(x) \infty \frac{1}{n!} \left(\frac{x}{2}\right)^n, \quad K_n(x) \infty \frac{(n-1)!}{2} \left(\frac{2}{x}\right)^n \quad (n \to \infty) \tag{3.2}$$

hold for the modified Bessel functions and Macdonald functions for finite values of the argument and large values of the index, we arrive at the deduction that an M_7 and $\eta > 0$ exist such that absolute values of all elements of the matrices mentioned can have the upper bound $M_7 m^\eta$.

Let us estimate the absolute values of the coefficients P_{mnij}. Taking account of the evenness (oddness) property of the functions $B_{ij}(t)$ noted in Sect. 2, we replace the integration limits in the penultimate formula in (1.4) by 0, ∞ and $\exp(-nt)$ by $\exp(-nt) \mp \exp nt$ by selecting the minus sign for $j = 1$ and the plus for $j = 2, 3$ (the sign selection rule is conserved in the expressions for $Q_{mij}^{\mp}$).

The following estimates $(\forall t \in \mathbf{R}^+)$ hold:

$$\exp(-\gamma h R_{ij}) < M_8 \exp(-\gamma h R_{jj}) \tag{3.3}$$

$$|e^{-nt} \mp e^{nt}| < 2e^{nt}, \quad |Q_{mij}^{\mp}| < M_9 (\zeta_j/\zeta_i)^m e^{mt}$$

Indeed the first of them follows from the limit relationship

$$\lim_{t \to \infty} \frac{\exp(-\gamma h R_{ij})}{\exp(-\gamma h R_{jj})} = 1$$

the second is obvious, and the third results from the inequality

$$|Q^{\mp}_{mij}| < 2(R_{ij}+\alpha_j)^m \zeta_i^{-m}, \quad t \in \mathbf{R}^+ \tag{3.4}$$

and the limit relationship

$$\lim_{t\to\infty} \frac{R_{ij}+\alpha_j}{e^t} = \zeta_j$$

Taking account of the inequalities (3.3) and (2.9) and using the integral representation for the Macdonald function, we obtain

$$|P_{mnij}| < 2M_{10}(\zeta_j/\zeta_i)^m K_{m+n+2q}(2\zeta_j\gamma h) \tag{3.5}$$

It follows from an analysis of relationships (1.9), (1.4), (3.5) and the above discussion that it is sufficient to prove convergence of a double series of the following configuration (μ is some positive number):

$$S_1 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} m^{\mu}(\zeta_j/\zeta_i)^m K_{m+n+2q}(2\zeta_j\gamma h)\frac{I_m(\zeta_i\gamma R)}{K_n(\zeta_j\gamma R)} \tag{3.6}$$

We will use the following estimates resulting from the equivalence relationships (3.2):

$$K_{m+n+2q}(2\zeta_j\gamma h) < M_{11}(m+n+2q-1)!/(\zeta_j\gamma h)^{m+n+2q} \tag{3.7}$$

$$K_n(\zeta_j\gamma R) > M_{12}(n-1)!\,2^n(\zeta_j\gamma R)^{-n}, \quad I_m(\zeta_i\gamma R) < M_{13}2^{-m}(\zeta_i\gamma R)^m/m!$$

Using the inequality

$$(m+n+2q-1)! < 2^{m+n+2q-1}(n-1)!\,(m+2q)!$$

and relationships (3.6) and (3.7), we can write

$$S_1 < M_{14}\sum_{m=0}^{\infty}(m+1)(m+2)\ldots(m+2q)\,m^{\mu}\left(\frac{R}{h}\right)^m \sum_{n=0}^{\infty}\left(\frac{R}{h}\right)^n$$

The series in m and n converge by virtue of the D'Alembert limit criterion (since $R/h<1$). Therefore, the series (3.1) also converges, i.e., $\Delta(\varepsilon,\gamma R)$ is a determinant of normal type.

Therefore, the method in /1/ for investigating fibre stability in a semi-infinite matrix near a free surface has been given a foundation.

REFERENCES

1. GUZ A.N. and LAPUSTA YU.N., Stability of a fibre near a free surface. Prikl. Mekhan., 22, 8, 1986.
2. GUZ A.N., On the construction of a stability theory for unidirectional fibrous materials, Prikl. Mekhan., 5, 2, 1969.
3. GUZ A.N., Stability of Elastic Bodies under Finite Deformations, Naukova Dumka, Kiev, 1973.
4. GUZ A.N., Stability of Three-dimensional Deformable Bodies, Naukova Dumka, Kiev, 1971.
5. GREGORY R.D., The propagation of waves in an elastic halfspace containing a cylindrical cavity, Proc. Cambridge Phil. Soc., 67, 3, 1970.
6. BABICH I.YU., GARASHCHUK I.N. and GUZ A.N., Stability of fibres in an elastic matrix for an inhomogeneous subcritical state, Prikl. Mekhan., 19, 11, 1983.
7. GUZ A.N., Principles of the Three-dimensional Theory of the Stability of Deformable Bodies, Vishcha Shkola, Kiev, 1986.

Translated by M.D.F.